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(Article begins on next page)



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GLOBAL L^p CONTINUITY OF FOURIER INTEGRAL OPERATORS

SANDRO CORIASCO AND MICHAEL RUZHANSKY

ABSTRACT. In this paper we establish global $L^p(\mathbb{R}^n)$ -regularity properties of Fourier integral operators. The orders of decay of the amplitude are determined for operators to be bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, as well as to be bounded from Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. This extends local L^p regularity properties of Fourier integral operators, as well as results of global $L^2(\mathbb{R}^n)$ boundedness, to the global setting of $L^p(\mathbb{R}^n)$. Global boundedness in weighted Sobolev spaces $W_s^{\sigma,p}(\mathbb{R}^n)$ is also established, and applications to hyperbolic partial differential equations are given.

1. INTRODUCTION

In this paper we investigate global $L^p(\mathbb{R}^n)$ continuity properties of non-degenerate Fourier integral operators. In particular, we are interested in the question of what decay properties of the amplitude guarantee the global boundedness of Fourier integral operators from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

The analysis of the local L^2 boundedness of Fourier integral operators goes back to Eskin [15] and Hörmander [16], who showed that non-degenerate Fourier integral operators with amplitudes in the symbol class $S_{1,0}^0$ are locally bounded on $L^2(\mathbb{R}^n)$. A Fourier integral operator of class $I^\mu(X, Y; \mathcal{C})$ is called non-degenerate if its canonical relation \mathcal{C} is locally a graph of a symplectic mapping from $T^*X \setminus 0$ to $T^*Y \setminus 0$. If the canonical relation of the operator degenerates, the local L^2 boundedness of zero order operators is known to fail, see e.g. Hörmander [18]. In this paper we will be concerned with non-degenerate operators only.

Since '70s this local L^2 boundedness result has been extended in different directions. On one hand, global $L^2(\mathbb{R}^n)$ boundedness has been studied, motivated by applications in microlocal analysis and hyperbolic partial differential equations. On the other hand, its extension to L^p spaces with $p \neq 2$ has been also under study motivated by applications in harmonic analysis.

The question of the global $L^2(\mathbb{R}^n)$ boundedness has been first widely investigated in the case of pseudo-differential operators. The phase is trivial in this case, so the main question is to determine minimal assumptions on the amplitude which guarantees the global $L^2(\mathbb{R}^n)$ boundedness. For example, one wants to relax an assumption that the symbol of a pseudo-differential operator is in the symbol class $S_{0,0}^0$ for operators to be still bounded on $L^2(\mathbb{R}^n)$. There are different sets of assumptions, see

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e.g. Calderón and Vaillancourt [5], Childs [6], Coifman and Meyer [7], Cordes [10], Sugimoto [35], etc. The question of the global $L^2(\mathbb{R}^n)$ boundedness of Fourier integral operators is more subtle, and involves different sets of assumptions on both phase and amplitude. Operators arising in applications to hyperbolic equations and Feynman path integrals have been considered e.g. in Asada [1], Asada and Fujiwara [2], Kumano-go [19], Boulkhemair [4]. On the other hand, applications to smoothing estimates for evolution partial differential equations require less restrictive assumptions on the phase, and the required estimates have been established by Ruzhansky and Sugimoto [28, 29].

Local L^p boundedness of Fourier integral operators has been under intensive study as well. In the case of $p \neq 2$ there is a loss of derivatives in L^p -spaces. For example, a loss of $(n-1)|1/p - 1/2|$ derivatives has been established for operators appearing as solutions to the wave equations, see e.g. Beals [3], Peral [24], Miyachi [23]. Finally, Seeger, Sogge and Stein [32] showed that general non-degenerate Fourier integral operators in the class $I^\mu(\mathbb{R}^n, \mathbb{R}^n; \mathcal{C})$ are locally bounded in $L^p(\mathbb{R}^n)$ provided that their amplitudes are in the class $S_{1,0}^\mu$ with $\mu \leq -(n-1)|1/p - 1/2|$, $1 < p < \infty$ (see also Sogge [33] and Stein [34]). In the case of $p = 1$, they showed that operators of order $\mu = -(n-1)/2$ are locally bounded from the Hardy space H^1 to L^1 , while Tao [37] showed that operators of the same order are also locally of weak type $(1,1)$. Extensions of these results with smaller loss of regularity under additional geometric assumptions on the canonical relations have been studied by Ruzhansky [26, 27]. There is also a result by Sugimoto [36] establishing global L^p estimates for translation invariant operators with phases with strictly convex level sets.

The aim of this paper is to establish global $L^p(\mathbb{R}^n)$ boundedness of Fourier integral operators, which depends on the growth/decay order of the amplitude in x and y variables. The results of this paper will extend the local L^p results of Seeger, Sogge and Stein [32] as well as global L^2 results of Asada and Fujiwara [2], Coriasco [12], and Ruzhansky and Sugimoto [28], to the global setting of $L^p(\mathbb{R}^n)$. In fact, for $p \neq 2$, we will observe that there is a loss not only of derivatives but also of growth/decay dependent on the value of p . Both of these losses disappear in the case $p = 2$. Consequently, using the global calculi of Fourier integral operators developed by Coriasco [12] and, more generally, by Ruzhansky and Sugimoto [30, 31], we can also obtain global weighted estimates in Sobolev spaces $W_\sigma^{s,p}(\mathbb{R}^n)$.

The L^p -continuity in Theorem 2.2 will be proved by interpolation. The end-points will be the L^2 -boundedness from Asada and Fujiwara [2] under assumption (2a) and from Ruzhansky and Sugimoto [28] under assumptions (2b) and (2c), and the $H^1 - L^1$ boundedness which is the main task of this paper.

We will be initially concerned with operators \mathcal{T} of the form

$$(1.1) \quad (\mathcal{T}u)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i[\langle x, \xi \rangle - \varphi(y, \xi)]} b(x, y, \xi) u(y) dy d\xi,$$

where φ is a real-valued phase function, positively homogeneous of order one in ξ , and b is an amplitude. Local L^p properties of such operators were considered by Seeger, Sogge and Stein [32] and their global L^2 properties were analysed by Ruzhansky and Sugimoto [28]. We note that a general Hörmander's Fourier integral operator can be always written in the form (1.1) microlocally while there are in general topological

obstructions globally. The microlocal qualitative properties of such operators are well-known, see e.g. Hörmander [16, 18] or Duistermaat [14]. Since the aim of this paper is to investigate L^p properties rather than trivialisations of the Maslov index, we will treat operators that can be written in the form (1.1) globally. We note that operators (1.1) and their adjoints appear as propagators to hyperbolic partial differential equations as well as canonical transforms in smoothing problems. Applications to hyperbolic Cauchy problems are given in Section 5.

Subsequently, we will deal with Fourier integral operators of the form

$$(1.2) \quad Au(x) = \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi) \widehat{u}(\xi) d\xi,$$

where φ is as above and the amplitude a does not depend on y .

Finally, we mention that results on the local L^p boundedness of Fourier integral operators with complex-valued phase functions have been established by Ruzhansky [27], extending previous local L^2 results by Melin and Sjöstrand [22] and Hörmander [17], and that there are also results in $(\mathcal{F}L^p)_{comp}$ spaces and in modulation spaces by Cordero, Nicola and Rodino [8].

Constants in this paper will be denoted by letters C and their values may vary even in the same formula. If the value of a constant is important and unchanged in a calculation, we will use sub-indices, denoting it e.g. by C_1, C_2 , etc. We will denote $\langle x \rangle = (1 + |x|^2)^{1/2}$. Occasionally, for functions $f(x, y, \xi, w), g(x, y, \xi, w)$, $x, y, \xi \in \mathbb{R}^n$, and w varying in a suitable parameter space, we will write $f \prec g$, $f \succ g$, if there exist constants $A, B > 0$ independent of w such that, for arbitrary x, y, ξ, w , we have $|f(x, y, \xi, w)| \leq A|g(x, y, \xi, w)|$, $|f(x, y, \xi, w)| \geq B|g(x, y, \xi, w)|$, respectively. If both $f \prec g$ and $f \succ g$ hold, we will write $f \sim g$. By $B_R(y)$ we will denote an open ball with radius R centred at y .

2. MAIN RESULTS

Let operator \mathcal{T} be given by

$$(2.1) \quad (\mathcal{T}u)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i[\langle x, \xi \rangle - \varphi(y, \xi)]} b(x, y, \xi) u(y) dy d\xi,$$

with a real-valued phase φ and an amplitude b which are *admissible*, in the sense of Definition 2.1 below. In particular, the phase φ must satisfy suitable smoothness conditions and

$$(2.2) \quad \begin{aligned} |\det \partial_y \partial_\xi \varphi(y, \xi)| &\geq C > 0, \quad \partial_y^\alpha \varphi(y, \xi) \prec \langle y \rangle^{1-|\alpha|} |\xi| \text{ for all } \alpha, \\ \langle \nabla_\xi \varphi(y, \xi) \rangle &\sim \langle y \rangle, \quad \langle d_y \varphi(y, \xi) \rangle \sim \langle \xi \rangle, \end{aligned}$$

either for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n \setminus 0$, or on $\text{supp } b$, depending on the other properties which are assumed in Definition 2.1.

Definition 2.1. *We say that the phase function φ and the amplitude function b are admissible if $|\xi| \geq \varepsilon$ on $\text{supp } b$, for some $\varepsilon > 0$, and, moreover, the following conditions hold:*

- (1) $\varphi \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$ is real-valued and positively homogeneous of order 1 in ξ on $\text{supp } b$, i.e. that $\varphi(y, \tau\xi) = \tau\varphi(y, \xi)$ for all $\tau > 0$ and $\xi \neq 0$, on $\text{supp } b$.
- (2) one of the following properties holds true:

(2a) let φ satisfy (2.2) for all $y \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n \setminus 0$ and be such that

$$(2.3) \quad \partial_y^\alpha \partial_\xi^\beta \varphi(y, \xi) \prec 1$$

for all x, y, ξ on $\text{supp } b$ and all multi-indices α, β such that $|\alpha + \beta| \geq 2$; in this part we also assume $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and let $b \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ satisfy

$$(2.4) \quad \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b(x, y, \xi) \prec \langle x \rangle^{m_1} \langle y \rangle^{m_2} \langle \xi \rangle^{\mu - |\gamma|}$$

for all $x, y, \xi \in \mathbb{R}^n$ and all multi-indices α, β, γ , with some $m_1, m_2 \in \mathbb{R}$ such that $m_1 + m_2 = m$;

(2b) let φ satisfy (2.2) on $\text{supp } b$, and

$$(2.5) \quad \partial_y^\alpha \partial_\xi^\beta \varphi(y, \xi) \prec 1$$

for all x, y, ξ on $\text{supp } b$ and all α, β such that $|\alpha| \geq 1$ and $|\beta| \geq 1$; moreover, let $b \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ satisfy

$$(2.6) \quad \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b(x, y, \xi) \prec \langle x \rangle^{m_1 - |\alpha|} \langle y \rangle^{m_2} \langle \xi \rangle^{\mu - |\gamma|}$$

for all $x, y, \xi \in \mathbb{R}^n$ and all multi-indices α, β, γ , with some $m_1, m_2 \in \mathbb{R}$ such that $m_1 + m_2 = m$;

(2c) let φ satisfy (2.2) on $\text{supp } b$, and

$$(2.7) \quad \partial_y^\alpha \partial_\xi^\beta \varphi(y, \xi) \prec \langle y \rangle^{1 - |\alpha|}$$

for all x, y, ξ on $\text{supp } b$ and all α, β such that $|\beta| \geq 1$; moreover, let $b \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ satisfy

$$(2.8) \quad \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b(x, y, \xi) \prec \langle x \rangle^{m_1} \langle y \rangle^{m_2 - |\beta|} \langle \xi \rangle^{\mu - |\gamma|}$$

for all $x, y, \xi \in \mathbb{R}^n$ and all multi-indices α, β, γ , with some $m_1, m_2 \in \mathbb{R}$ such that $m_1 + m_2 = m$.

The main result of this paper is the following:

Theorem 2.2. *Let $1 < p < \infty$ and $m, \mu \in \mathbb{R}$. Let \mathcal{T} be operator (2.1) and assume that the phase function φ and the amplitude b are admissible, in the sense of Definition 2.1. Then \mathcal{T} extends to a bounded operator from $L^p(\mathbb{R}^n)$ to itself, provided that*

$$(2.9) \quad m \leq -n \left| \frac{1}{p} - \frac{1}{2} \right| \quad \text{and} \quad \mu \leq -(n-1) \left| \frac{1}{p} - \frac{1}{2} \right|.$$

Let us now discuss the assumptions of Theorem 2.2. First of all, we note that assumptions (2.2) are very natural in the sense that they ask that φ is essentially of order one in both y and ξ . Condition

$$(2.10) \quad |\det \partial_y \partial_\xi \varphi(y, \xi)| \geq C > 0,$$

for all $y \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n \setminus 0$ is simply a global version of the local graph condition of the non-degeneracy of Fourier integral operator (2.1). Assumption (2.4) says that b has a symbolic behaviour in ξ and is of order $m_1 + m_2 = m$ jointly in x and y .

We assume that $\xi \neq 0$ on the support of b to avoid the singularity of the (homogeneous) phase at the origin. We note that this issue does not arise in local boundedness

problems (as in [32]) since the corresponding part of the operator is locally smoothing. In our situation it is still smoothing but may, in principle, destroy the global behaviour with respect to x and y . Some global results in $L^2(\mathbb{R}^n)$ for small frequencies have been established by Ruzhansky and Sugimoto in [28] using weighted estimates for multipliers of Kurtz and Wheeden [21], and we refer to [28] for a discussion of complications that arise in this situation.

Assumption (2b) is different from (2a) in that we do not assume the boundedness (2.3), but assume the boundedness only of mixed derivatives (i.e. with $|\alpha| \geq 1$ and $|\beta| \geq 1$), and in addition assume that derivatives of b have some decay properties in (2.6) or in (2.8). In assumption (2c) we also allow non-mixed derivatives (i.e. ∂_ξ^β -derivatives when $\alpha = 0$) to grow in y . Moreover, in both (2b) and (2c) we assume (2.2) to hold only on the support of b .

We note that propagators for hyperbolic partial differential equations lead to operators (2.1) with $b(x, y, \xi) = b(y, \xi)$ independent of x , in which case assumption (2.6) becomes trivial if $\alpha \neq 0$. For these propagators also the boundedness (2.3) is satisfied under natural assumptions on the symbol of the hyperbolic equation. However, we do not always want to assume the boundedness (2.3) since it fails for non-mixed derivatives (i.e. when $\alpha = 0$ or $\beta = 0$), e.g. in applications to smoothing estimates for dispersive equations. For example, it is shown in [28, 29] that for canonical transforms appearing there condition (2.3) fails, but it is also shown that additional decay of derivatives as in (2.6) or (2.8) holds.

If the amplitude b in Theorem 2.2 is compactly supported in (x, y) , Theorem 2.2 implies the local L^p boundedness under the assumptions in Seeger, Sogge and Stein [32], implying, in particular, that the order μ in Theorem 2.2 cannot be improved in general. Let us now give some explanation about the order m . In [8], Cordero, Nicola and Rodino investigated the question of the boundedness of Fourier integral operators on $(\mathcal{F}L^p(\mathbb{R}^n))_{comp}$, the space of compactly supported distributions for which the Fourier transform is in $L^p(\mathbb{R}^n)$. They proved that if the amplitude of an operator is of order $-n \left| \frac{1}{p} - \frac{1}{2} \right|$ in ξ (plus additional assumptions), then the operator is continuous on $(\mathcal{F}L^p(\mathbb{R}^n))_{comp}$. They also showed that this order of decay is sharp by constructing a counterexample for higher orders. Roughly speaking, the conjugation with the Fourier transform interchanges the roles of x and ξ , so the orders in [8] correspond to orders $m = -n \left| \frac{1}{p} - \frac{1}{2} \right|$ and $\mu = -\infty$ for operators in the setting of Theorem 2.2 since the assumption of the compact support in $(\mathcal{F}L^p(\mathbb{R}^n))_{comp}$ corresponds to locally smoothing operators in $L^p(\mathbb{R}^n)$. From this point of view, our Theorem 2.2 also improves the result of [8] with respect to μ to the order $\mu = -(n-1) \left| \frac{1}{p} - \frac{1}{2} \right|$, which cannot be improved further in general. However, the order m in Theorem 2.2 can still be improved if we restrict the size of the support while still allowing it to move to infinity. In this case a uniform estimate is possible for $m \leq -(n-1) \left| \frac{1}{p} - \frac{1}{2} \right|$ and it is given in Theorem 2.5. The same improved threshold for the order m can be achieved for the Fourier integral operators (1.2), as stated in Theorem 2.6.

To prove Theorem 2.2 we use interpolation between the $L^2(\mathbb{R}^n)$ -boundedness and boundedness from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. The global $L^2(\mathbb{R}^n)$ -boundedness under assumptions (2a) and (2b)–(2c) would follow from the results of Asada

and Fujiwara [2] and Ruzhansky and Sugimoto [28], respectively. Thus, the main point is to prove the boundedness from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. This can be achieved by using the atomic decomposition of $H^1(\mathbb{R}^n)$ and splitting the argument for atoms with large and small supports. However, there is a number of difficulties in this argument compared with that of [32]. For example, supports are no longer bounded and can become very large, and hence, while this case is simple for the local boundedness, it requires to be analysed further in the global setting. Another global feature is that even if the supports of atoms may be small, they may still move to infinity (while remaining small). We deal with this situation by introducing a dyadic decomposition in frequency which depends on y . The dyadic pieces that we work with are of the size 2^{-k} in the radial direction and of the size $2^{-\frac{k}{2}}\langle y \rangle^{\frac{1}{2}}$ in other directions (tangential to the sphere in the frequency space). Thus, we obtain the following theorem in the setting of the Hardy space $H^1(\mathbb{R}^n)$:

Theorem 2.3. *Let \mathcal{T} be the Fourier integral operator (2.1). Under the hypotheses of Theorem 2.2, operator \mathcal{T} extends to a bounded operator from the Hardy space $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, provided that $m \leq -n/2$ and $\mu \leq -(n-1)/2$.*

We can establish also a result in weighted Sobolev spaces. Let $W_s^{\sigma,p}(\mathbb{R}^n)$ denote the weighted Sobolev space, i.e. the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\langle x \rangle^s (1 - \Delta)^{\sigma/2} f(x)$ belongs to $L^p(\mathbb{R}^n)$.

Theorem 2.4. *Let $1 < p < \infty$ and let $\sigma, s \in \mathbb{R}$. Let \mathcal{T} be the Fourier integral operator (2.1) as in Theorem 2.2 with orders $m, \mu \in \mathbb{R}$, and let $m_p = -n \left| \frac{1}{p} - \frac{1}{2} \right|$, $\mu_p = -(n-1) \left| \frac{1}{p} - \frac{1}{2} \right|$. Then operator \mathcal{T} extends to a bounded operator from $W_s^{\sigma,p}(\mathbb{R}^n)$ to $W_{s-m-m_p}^{\sigma-\mu-\mu_p,p}(\mathbb{R}^n)$.*

Theorem 2.4 follows from Theorem 2.2 and composition formulae of Fourier integral operators with pseudo-differential operators as in [30] or in [31]. In fact, here we only need a special class of pseudo-differential operators, namely of operators with symbols $\pi_{s,\sigma}(x, \xi) = \langle x \rangle^s \langle \xi \rangle^\sigma$ for which we have $(\text{Op } \pi_{s,\sigma})(W_s^{\sigma,p}(\mathbb{R}^n)) = L^p(\mathbb{R}^n)$. Global composition formulae of [30, 31] will be also used in the proof of Theorem 2.3.

The assumptions on the order of the amplitude in Theorem 2.2 can be relaxed if we work with functions with compact support. We will assume that the supports are uniformly bounded but will still allow them to move to infinity (while remaining bounded). In this situation the proof of Theorem 2.2 will also imply the following

Theorem 2.5. *Let $1 < p < \infty$ and let $m, \mu \in \mathbb{R}$. Let \mathcal{T} be the Fourier integral operator (2.1) as in Theorem 2.2. Let $R > 0$. Let $\mathcal{V}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ be the set of all functions $f \in L^p(\mathbb{R}^n)$ for which there exists $y \in \mathbb{R}^n$ such that $\text{supp } f \subset B_R(y)$, and let $\mathcal{V}(\mathbb{R}^n)$ have the topology induced by $L^p(\mathbb{R}^n)$. Then operator \mathcal{T} extends to a continuous operator from $\mathcal{V}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, provided that*

$$(2.11) \quad m \leq -(n-1) \left| \frac{1}{p} - \frac{1}{2} \right| \quad \text{and} \quad \mu \leq -(n-1) \left| \frac{1}{p} - \frac{1}{2} \right|.$$

Theorem 2.5 will follow from Remarks 3.3 and 3.7. We also have natural counterparts of Theorem 2.5 for H^1 and $W_s^{\sigma,p}$ as in Theorems 2.3 and 2.4.

We note that for large atoms the cancellation condition is not used in our proof, so that the results extend to the boundedness from the local Hardy space $H_{loc}^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. The order in x is improved in Theorem 2.5 and in Theorem 2.6.

By an argument similar to the one used in [9], it is also possible to prove the L^p -continuity of the classes of Fourier integral operators considered in [12], where the phase function is assumed positively homogeneous of order 1 in ξ and satisfies (2.2):

Theorem 2.6. *Let $A = A_{\varphi,a}$ be a Fourier integral operator of the form*

$$(2.12) \quad Au(x) = \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} a(x,\xi) \widehat{u}(\xi) d\xi,$$

with a real-valued phase function $\varphi \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ such that $\varphi(x, \tau\xi) = \tau\varphi(x, \xi)$ for all $\tau > 0$ and $\xi \neq 0$, and assume that the condition (2.2) holds true for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. Moreover, assume that $|\xi| \geq \varepsilon$, for some $\varepsilon > 0$, on the support of the amplitude a , and that $a \in S^{m,\mu}$, i.e. that

$$\partial_x^\alpha \partial_\xi^\beta a(x, \xi) \prec \langle x \rangle^{m-|\alpha|} \langle \xi \rangle^{\mu-|\beta|},$$

for all $x, \xi \in \mathbb{R}^n$ and all multi-indices α, β , with some $m, \mu \in \mathbb{R}$. Then A extends to a bounded operator from $L^p(\mathbb{R}^n)$ to itself, provided that

$$(2.13) \quad m \leq -(n-1) \left| \frac{1}{p} - \frac{1}{2} \right| \quad \text{and} \quad \mu \leq -(n-1) \left| \frac{1}{p} - \frac{1}{2} \right|.$$

The thresholds (2.13) are sharp, by a modification of a counterexample described in [9]. The improvement in Theorem 2.6 compared to that in Theorem 2.2, (2c), comes from the independence of the amplitude of A on y -variable, if we write the adjoint A^* in the form of an operator \mathcal{T} in Theorem 2.2. The proof of Theorem 2.6 is given in Section 4. Finally, the composition formulae in [12] together with Theorem 2.6 imply the analog of Theorem 2.4 for the operator A :

Theorem 2.7. *Let $1 < p < \infty$ and let $\sigma, s \in \mathbb{R}$. Let A be the Fourier integral operator (1.2) as in Theorem 2.6 with orders $m, \mu \in \mathbb{R}$, and let $m_p = -(n-1) \left| \frac{1}{p} - \frac{1}{2} \right|$. Then operator A extends to a bounded operator from $W_s^{\sigma,p}(\mathbb{R}^n)$ to $W_{s-m-m_p}^{\sigma-\mu-m_p,p}(\mathbb{R}^n)$.*

We finish by briefly indicating the example for the sharpness of order m in Theorem 2.6. Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth diffeomorphism whose restriction to $(0, 1)$ is a non-linear diffeomorphism on $(0, 1)$, i.e. there exists an interval $I \subset (0, 1)$ such that $|\psi''| > 0$ on the closure of I . Moreover, assume that $0 < c \leq |\psi'(t)| \leq C$ for all $t \in \mathbb{R}$. Define

$$\phi(\xi) := (\psi(\xi_1/\xi_n), \dots, \psi(\xi_{n-1}/\xi_n)) \xi_n$$

microlocally in a narrow cone around $(0, 1)$, i.e. for $|\xi_n| \geq C|\xi'|$, where $\xi' = (\xi_1, \dots, \xi_{n-1})$. Let $G \in C_0^\infty(\mathbb{R}^n)$ be such that $G \geq 0$, $G(\xi', \xi_n) = 1$ for $\epsilon \leq |\xi_n| \leq 1$, and such that the support of G is contained in the set (ξ', ξ_n) with $\epsilon/2 \leq |\xi_j| \leq M$, $j = 1, \dots, n-1$, $M \geq |\xi_n| \geq C|\xi'|$, for some sufficiently small $\epsilon > 0$ and some $M > 0$. Let us write $y = (y', y_n)$. Define the operator

$$\mathcal{T}f(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i[\langle x, \xi \rangle - \langle y', \phi(\xi) \rangle - y_n \xi_n]} \langle y \rangle^m G(\xi) \chi_{[1,2]}(y_n) f(y) dy d\xi,$$

where $\chi_{[1,2]} \in C_0^\infty(\mathbb{R})$ is a non-zero function with $\text{supp } \chi_{[1,2]} \subset [1, 2]$. We notice that \mathcal{T} is a locally smoothing operator with phase and amplitude satisfying conditions of Theorem 2.2. After conjugating \mathcal{T} with the Fourier transform and factoring out ξ_n , we observe that $y \sim y'$ on the support of the amplitude of \mathcal{T} . Therefore, if \mathcal{T} is bounded on $L^p(\mathbb{R}^n)$, the operator $\mathcal{F}^{-1} \circ \mathcal{T} \circ \mathcal{F}$ with ξ_n, y_n factored out, would be bounded on $(\mathcal{F}L^p(\mathbb{R}^{n-1}))_{\text{comp}}$. Since this is an operator of the form considered in Section 6.2 in [8], we must have $m \leq -(n-1)|1/p - 1/2|$, $1 \leq p \leq \infty$.

3. PROOF OF THEOREM 2.3

Since Theorem 2.2 follows by complex interpolation from Theorem 2.3 and L^2 -boundedness results in [2] and [28] under assumptions (2a) and (2b)–(2c), respectively, we need to prove Theorem 2.3. This will be achieved through various subsequent steps. We also note that on several occasions we will use the composition formulae for globally defined Fourier integral operators and pseudo-differential operators. In order not to list the corresponding lengthy conditions for the calculus for such operators to work, we refer to Theorems 2.1, 2.5 and 2.8 in [31] for exact formulations.

Given $f \in H^1(\mathbb{R}^n)$, we can decompose (see e.g. [34]) function

$$f = \sum_Q \lambda_Q a_Q,$$

where

$$\sum_Q |\lambda_Q| \simeq \|f\|_{H^1(\mathbb{R}^n)},$$

and the atoms $a_Q \in H^1(\mathbb{R}^n)$ have the following properties:

- (1) $\text{supp } a_Q \subset Q$, where $Q \subset \mathbb{R}^n$ is a cube of sidelength q ;
- (2) $\|a_Q\|_{L^\infty(\mathbb{R}^n)} \leq |Q|^{-1}$;
- (3) $\int_Q a_Q(y) dy = 0$.

Theorem 2.3 would then follow if we show that

$$(3.1) \quad \|\mathcal{T}a_Q\|_{L^1(\mathbb{R}^n)} \leq C,$$

for a constant C independent of a_Q .

Let $F = F(x, y)$ denote the distribution kernel of \mathcal{T} , given by the oscillatory integral

$$(3.2) \quad F(x, y) = \int_{\mathbb{R}^n} e^{i[\langle x, \xi \rangle - \varphi(y, \xi)]} b(x, y, \xi) d\xi.$$

We begin showing that the amplitude function can be assumed supported only in a suitable neighbourhood of the wave front set of the distributional kernel of \mathcal{T} :

Proposition 3.1. *Let $\chi = \chi(x, y, \xi)$ be supported in*

$$E_k = \{(x, y, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : |x - \nabla_\xi \varphi(y, \xi)| \leq k\langle x \rangle\},$$

for $k \in (0, 1)$ suitably small, and such that $\chi|_{E_{\frac{k}{2}}} \equiv 1$. Moreover¹, let us assume that χ (is smooth and) satisfies $S^{0,0,0}$ estimates on $\text{supp } b$, that is

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma \chi(x, y, \xi) \prec \langle x \rangle^{-|\alpha|} \langle y \rangle^{-|\beta|} \langle \xi \rangle^{-|\gamma|}.$$

We set $\tilde{b} = (1 - \chi)b$. Then, defining

$$(3.3) \quad \tilde{F}(x, y) = \int_{\mathbb{R}^n} e^{i[\langle x, \xi \rangle - \varphi(y, \xi)]} \tilde{b}(x, y, \xi) d\xi,$$

it follows that $\tilde{F} \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, which implies, in particular, that

$$(3.4) \quad \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \tilde{F}(x, y) a_Q(y) dy \right| dx \leq C,$$

with a constant C independent of a_Q .

Proof. We will show that kernel \tilde{F} satisfies

$$(3.5) \quad \partial_x^\alpha \partial_y^\beta \tilde{F}(x, y) \prec (\langle x \rangle \langle y \rangle)^{-N},$$

for all $N \in \mathbb{N}$, $x, y \in \mathbb{R}^n$ and all multi-indices α, β . By the hypotheses on b and φ , it is clear that it is enough to prove the estimate only for $\alpha = \beta = 0$ and arbitrary order in x, y, ξ for \tilde{b} .

Indeed, we have $|x - \nabla_\xi \varphi(y, \xi)| \succ \langle x \rangle$ on $\text{supp } \tilde{b}$, so that the operator L_ξ , acting on functions $v = v(x, y, \xi)$ with respect to ξ as

$$(L_\xi v)(x, y, \xi) = \sum_{j=1}^n i \partial_{\xi_j} \left(\frac{x_j - \partial_{\xi_j} \varphi(y, \xi)}{|x - \nabla_\xi \varphi(y, \xi)|^2} v(x, y, \xi) \right),$$

is well-defined on $\text{supp } \tilde{b}$. Moreover, on $\text{supp } \tilde{b}$, we have

$$|x - \nabla_\xi \varphi(y, \xi)| \succ \langle x \rangle.$$

Then

$$|\nabla_\xi \varphi(y, \xi)| \leq |x - \nabla_\xi \varphi(y, \xi)| + |x| \prec |x - \nabla_\xi \varphi(y, \xi)|,$$

and it follows that we also have

$$|x - \nabla_\xi \varphi(y, \xi)| \succ \langle \nabla_\xi \varphi(y, \xi) \rangle \succ \langle y \rangle.$$

Now (3.5) follows by integrating by parts in (3.3), observing that

$${}^t L_\xi e^{i[\langle x, \xi \rangle - \varphi(y, \xi)]} = e^{i[\langle x, \xi \rangle - \varphi(y, \xi)]}.$$

Then (3.4) holds, since, for all $N \in \mathbb{N}$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \tilde{F}(x, y) a_Q(y) dy \right| dx &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\tilde{F}(x, y)| |a_Q(y)| dy dx \\ &\leq \tilde{C} \int_{\mathbb{R}^n} \langle x \rangle^{-N} dx \int_{\mathbb{R}^n} |a_Q(y)| dy \leq C |Q| |Q|^{-1} = C. \end{aligned}$$

¹With $h \in C^\infty(\mathbb{R})$ such that $h|_{(-\infty, \frac{1}{2})} \equiv 1$ and $h|_{(1, +\infty)} \equiv 0$, $k \in (0, 1)$, set

$$\chi(x, y, \xi) = h\left(\frac{|x - \nabla_\xi \varphi(y, \xi)|}{k \langle x \rangle}\right).$$

□

Therefore, from now on we can then assume that for some $k \in (0, 1)$ we have

$$(3.6) \quad \text{supp } b \subseteq D = \{(x, y, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : |x - \nabla_\xi \varphi(y, \xi)| \leq k\langle x \rangle\}.$$

This implies that on $\text{supp } b$ we have

$$\langle x \rangle \sim \langle \nabla_\xi \varphi(y, \xi) \rangle \sim \langle y \rangle,$$

which in turn implies that $C_1 \langle y \rangle \leq \langle x \rangle \leq C_2 \langle y \rangle$, $x, y \in \mathbb{R}^n$, for suitable constants $C_1, C_2 > 0$.

Proposition 3.2. *Let a_Q be an atom in $H^1(\mathbb{R}^n)$, supported in a cube $Q \subset \mathbb{R}^n$ centred at $y_0 \in \mathbb{R}^n$ and with sidelength $q \geq 1$ (hence also $|Q| \geq 1$). Then, estimate (3.1) holds with a constant C independent of a_Q .*

Proof. Let us denote by M_s the multiplication operator $(M_s v)(x) = \langle x \rangle^s v(x)$. From composition formulae with pseudo-differential operators (see [31]) it follows that operator $M_{\frac{n}{2}} \mathcal{T}$ is then a Fourier integral operator with amplitude bounded in x and y , and of order $-\frac{n-1}{2}$ in ξ . Consequently, operator $M_{\frac{n}{2}} \mathcal{T}$ is bounded on $L^2(\mathbb{R}^n)$ by [2] under assumption (2a) and by [28] under assumptions (2b) and (2c). Applying Hölder's inequality and denoting

$$D_{q, y_0} = \{x \in \mathbb{R}^n \mid C_1 \langle y \rangle \leq \langle x \rangle \leq C_2 \langle y \rangle, y \in Q\},$$

we get

$$\begin{aligned} \|\mathcal{T} a_Q\|_{L^1(\mathbb{R}^n)} &= \int_{\substack{\langle x \rangle \sim \langle y \rangle \\ y \in Q}} |\langle x \rangle^{-\frac{n}{2}} (M_{\frac{n}{2}} \mathcal{T} a_Q)(x)| dx \\ &\leq \left(\int_{D_{q, y_0}} \langle x \rangle^{-n} dx \right)^{\frac{1}{2}} \| (M_{\frac{n}{2}} \mathcal{T}) a_Q \|_{L^2(\mathbb{R}^n)} \\ &\leq \tilde{C} \|a_Q\|_{L^2(\mathbb{R}^n)} \left[\int_{D_{q, y_0}} (1 + |x|^2)^{-\frac{n}{2}} dx \right]^{\frac{1}{2}} \\ &= \tilde{C} |Q|^{-1} |Q|^{\frac{1}{2}} \left[\int_{D_{q, y_0}} (1 + |x|^2)^{-\frac{n}{2}} dx \right]^{\frac{1}{2}} \\ &= \tilde{C} \left[|Q|^{-1} \int_{D_{q, y_0}} (1 + |x|^2)^{-\frac{n}{2}} dx \right]^{\frac{1}{2}} \leq C, \end{aligned}$$

where $C \geq 0$ does not depend on a_Q . Indeed, let us prove the boundedness of the expression in the last line. Let us set $A = 1 + \frac{|C_1^2 - 1|^{\frac{1}{2}}}{C_1}$. The required boundedness is a consequence of the following steps:

- choose $\psi \in \mathcal{C}^\infty(\mathbb{R})$ supported in $(-\infty, 2]$, taking values in $[0, 1]$, and such that $\psi(t) = 1$ for $t \in (-\infty, 1]$. Set $\chi(q, y_0) = \psi\left(\frac{|y_0|}{Aq\sqrt{n}}\right)$ and let

$$I_1 = \chi(q, y_0) |Q|^{-1} \int_{D_{q, y_0}} (1 + |x|^2)^{-\frac{n}{2}} dx,$$

$$I_2 = (1 - \chi(q, y_0)) |Q|^{-1} \int_{D_{q, y_0}} (1 + |x|^2)^{-\frac{n}{2}} dx;$$

- on the support of $\chi(q, y_0)$ we have $|y_0| \leq 2Aq\sqrt{n}$, so, for $x \in D_{q, y_0}$,

$$|x| < \langle x \rangle \leq C_2 \langle y \rangle \leq C_2 \sqrt{(|y - y_0| + |y_0|)^2 + 1}$$

$$\leq C_2 \sqrt{\left(\frac{q\sqrt{n}}{2} + 2Aq\sqrt{n}\right)^2 + 1} \leq Kq,$$

where $K > 0$ is independent of $q \geq 1$ and $y_0 \in \mathbb{R}^n$. Then, $D_{q, y_0} \subset B_{Kq}(0)$, where $B_{Kq}(0)$ is the ball centred at the origin with radius Kq , and we have

$$I_1 \leq |Q|^{-1} |B_{Kq}(0)| \leq K^n |B_1(0)| = B_1,$$

with $B_1 > 0$ independent of $q \geq 1$, $y_0 \in \mathbb{R}^n$;

- on the support of $1 - \chi(q, y_0)$ we have $|y_0| \geq Aq\sqrt{n} > 1$ and, for $x \in D_{q, y_0}$, we have

$$\sqrt{C_1^2 |y|^2 + C_1^2 - 1} \leq |x| \leq \sqrt{C_2^2 |y|^2 + C_2^2 - 1}, \quad y \in Q.$$

Note also that, on the support of $1 - \chi(q, y_0)$, for $y \in Q$ we have

$$\frac{|y - y_0|}{|y_0|} \leq \frac{q\sqrt{n}}{2} \frac{1}{Aq\sqrt{n}} = \frac{1}{2A} < \frac{1}{2}$$

and

$$|y| \geq |y_0| - |y - y_0| = |y_0| \left(1 - \frac{|y - y_0|}{|y_0|}\right) \geq |y_0| \left(1 - \frac{1}{2A}\right) > \frac{|y_0|}{2} > \frac{1}{2}.$$

Hence we can estimate

$$\begin{aligned} C_1^2 |y|^2 + C_1^2 - 1 &\geq |y_0|^2 \left[C_1^2 \left(1 - \frac{|y - y_0|}{|y_0|}\right)^2 + \frac{C_1^2 - 1}{|y_0|^2} \right] \\ &\geq |y_0|^2 \left[C_1^2 \left(1 - \frac{1}{2A}\right)^2 - \frac{|C_1^2 - 1|}{|y_0|^2} \right] \\ &\geq |y_0|^2 \left[\frac{C_1^2 (2A - 1)^2}{4A^2} - \frac{|C_1^2 - 1|}{A^2 q^2 n} \right] \\ &\geq |y_0|^2 \frac{C_1^2 q^2 n \left(1 + \frac{2|C_1^2 - 1|^{\frac{1}{2}}}{C_1}\right)^2 - 4|C_1^2 - 1|}{4A^2 q^2 n} \\ &\geq |y_0|^2 \frac{q^2 n (C_1^2 + 4C_1 |C_1^2 - 1|^{\frac{1}{2}})}{4A^2 q^2 n} > 0, \end{aligned}$$

from which we get that

$$r_1 := \min_{y \in Q} \sqrt{C_1^2 |y|^2 + C_1^2 - 1} \geq K_1 |y_0| > 0,$$

with $\frac{3C_1}{2} > K_1 > 0$ independent of $q \geq 1$, $y_0 \in \mathbb{R}^n$. Since $C_2 \geq C_1$, on the support of $1 - \chi(q, y_0)$ we have $C_2^2 |y|^2 + C_2^2 - 1 > 0$, and

$$\begin{aligned} \sqrt{C_2^2 |y|^2 + C_2^2 - 1} &\leq \sqrt{C_2^2 (|y_0| + |y - y_0|)^2 + C_2^2} \\ &\leq C_2 |y_0| \sqrt{\left(1 + \frac{|y - y_0|}{|y_0|}\right)^2 + \frac{1}{|y_0|^2}} \\ &\leq 2C_2 |y_0|, \end{aligned}$$

so that $r_1 < r_2 := \max_{y \in Q} \sqrt{C_2^2 |y|^2 + C_2^2 - 1} \leq K_2 |y_0|$ with $K_2 > K_1 > 0$ independent of $q \geq 1$, $y_0 \in \mathbb{R}^n$; we have then proved that, on the support of $1 - \chi(q, y_0)$, $D_{q, y_0} \subset \overline{B_{r_2}(0)} \setminus B_{r_1}(0)$, hence

$$\begin{aligned} I_2 &\leq (1 - \chi(q, y_0)) |Q|^{-1} |B_1(0)| \int_{r_1}^{r_2} \frac{r^{n-1}}{(1 + r^2)^{\frac{n}{2}}} dr \\ &\leq |B_1(0)| \int_{r_1}^{r_2} \frac{dr}{r} \leq |B_1(0)| \log \frac{K_2}{K_1} = B_2, \end{aligned}$$

with $B_2 > 0$ independent of $q \geq 1$, $y_0 \in \mathbb{R}^n$.

The proof is complete. \square

Remark 3.3. Let operator \mathcal{T} be as in Theorem 2.2 with μ satisfying (2.9) but with any $m \leq 0$. Let $R > 0$. Let a_Q be an atom in $H^1(\mathbb{R}^n)$, supported in a cube $Q \subset \mathbb{R}^n$ centred at $y_0 \in \mathbb{R}^n$ and with sidelength q such that $R \geq q \geq 1$. Then, estimate (3.1) holds with a constant C independent of such a_Q .

This remark follows immediately from the proof of Proposition 3.2 if we observe that the boundedness of I_1 is actually independent of the order of b in x , while the boundedness of I_2 is a consequence of the fact that the volume of D_{q, y_0} is bounded by a uniform constant for all cubes Q in Remark 3.3.

Of course, the argument in the proof of Proposition 3.2 still holds if the hypothesis $|Q| \geq 1$ is replaced by $|Q| \geq Q_0 > 0$, or, equivalently, by $q \geq q_0 > 0$. In the next steps of the proof we can then assume that a_Q is supported in a cube Q with sidelength $q = 2^{-j}$, $j \geq j_0$, where j_0 is chosen so large that $\frac{q}{2} \sqrt{n} < 1$. In this way,

$$y \in Q \implies |y - y_0| \leq \frac{q}{2} \sqrt{n} \implies \langle y \rangle \sim \langle y_0 \rangle,$$

with y_0 centre of Q , so that we also have, on $\text{supp } b$, that $\langle x \rangle \sim \langle y_0 \rangle$.

We now define an “exceptional set” set \mathcal{N}_Q , which covers

$$(3.7) \quad \Sigma = \{x = \nabla_\xi \varphi(y, \xi) \text{ for some } y \in Q, \xi \in \mathbb{R}^n\},$$

and use again L^2 -boundedness results, together with Hölder and Hardy-Littlewood-Sobolev inequalities, to estimate $\|\mathcal{T}a_Q\|_{L^1}$ on that set.

Choose unit vectors ξ_k^ν , $\nu = 1, \dots, N(k, y)$, $k \geq j_0$, $y \in \mathbb{R}^n$, such that:

- $|\xi_k^\nu - \xi_{k'}^{\nu'}| \geq C_0 2^{-\frac{k}{2}} \langle y \rangle^{-\frac{1}{2}}$, $\nu \neq \nu'$, for some fixed positive constant $C_0 < 1$;
- the unit sphere \mathbb{S}^{n-1} is covered by the balls centred at ξ_k^ν with radius $2^{-\frac{k}{2}} \langle y \rangle^{-\frac{1}{2}}$.

We have then

$$N(k, y) \approx 2^{k \frac{n-1}{2}} \langle y \rangle^{\frac{n-1}{2}}.$$

For $y \in Q$ and a constant M to be fixed later, define

$$(3.8) \quad \mathcal{R}_{k\nu}^y = \left\{ x : |\langle x - \nabla_\xi \varphi(y, \xi_k^\nu), \xi_k^\nu \rangle| \leq M 2^{-k} \text{ and } |\Pi_{k\nu}^\perp(x - \nabla_\xi \varphi(y, \xi_k^\nu))| \leq M 2^{-\frac{k}{2}} \langle y \rangle^{\frac{1}{2}} \right\},$$

where $\Pi_{k\nu}^\perp$ is the projection onto the plane orthogonal to ξ_k^ν . Set $\mathcal{R}_{k\nu}^y$ is then a n -rectangle with $n-1$ sides of length $M 2^{-\frac{k}{2}} \langle y \rangle^{\frac{1}{2}}$ and one side of length $M 2^{-k}$. If Q has sidelength $q = 2^{-j}$, $j \geq j_0$, we define

$$(3.9) \quad \mathcal{N}_Q = \bigcup_{y \in Q} \bigcup_{\nu=1}^{N(j,y)} \mathcal{R}_{j\nu}^y.$$

Since $|\mathcal{R}_{j\nu}^y| \approx 2^{-j \frac{n+1}{2}} \langle y_0 \rangle^{\frac{n-1}{2}}$ for $y \in Q$, it follows that

$$(3.10) \quad |\mathcal{N}_Q| \leq C 2^{j \frac{n-1}{2}} \langle y_0 \rangle^{\frac{n-1}{2}} 2^{-j \frac{n+1}{2}} \langle y_0 \rangle^{\frac{n-1}{2}} = C 2^{-j} \langle y_0 \rangle^{n-1} = C |Q|^{\frac{1}{n}} \langle y_0 \rangle^{n-1},$$

for some constant $C \geq 0$ independent of $j \geq j_0$, $y_0 \in \mathbb{R}^n$.

Lemma 3.4. *If in (3.8) we take*

$$M = \sup_{\substack{|\alpha|=2,3 \\ (y,\xi) \in \mathbb{R}^n \times \mathbb{R}^n}} \langle y \rangle^{-1} \langle \xi \rangle^{-1+|\alpha|} |\partial_\xi^\alpha \varphi(y, \xi)|,$$

the singular set Σ defined in (3.7) is a subset of \mathcal{N}_Q .

Proof. Let us denote

$$\text{vers}(\xi) = \frac{\xi}{|\xi|}.$$

Since, for all $\xi \in \mathbb{R}^n$, we have

$$|\text{vers}(\xi) - \xi_j^\nu| \leq 2^{-\frac{j}{2}} \langle y \rangle^{-\frac{1}{2}}$$

for some $\nu = 1, \dots, N(j, y)$, then, with M chosen as above, we have $\nabla_\xi \varphi(y, \xi) \in \mathcal{R}_{j\nu}^y$. Indeed, $\nabla_\xi \varphi(y, \xi)$ is homogeneous of order 0 in ξ and $\Pi_{j\nu}^\perp$ is a projection, so that

$$\begin{aligned} |\Pi_{j\nu}^\perp(\nabla_\xi \varphi(y, \xi) - \nabla_\xi \varphi(y, \xi_j^\nu))| &\leq |\nabla_\xi \varphi(y, \text{vers}(\xi)) - \nabla_\xi \varphi(y, \xi_j^\nu)| \\ &\leq M \langle y \rangle |\text{vers}(\xi) - \xi_j^\nu| \leq M 2^{-\frac{j}{2}} \langle y \rangle^{\frac{1}{2}}. \end{aligned}$$

Moreover, again in view of the homogeneity of the phase function, if we set

$$h_j^\nu(y, \xi) = \langle \nabla_\xi \varphi(y, \xi), \xi_j^\nu \rangle - \langle \nabla_\xi \varphi(y, \xi_j^\nu), \xi_j^\nu \rangle = \langle \nabla_\xi \varphi(y, \xi), \xi_j^\nu \rangle - \varphi(y, \xi_j^\nu),$$

we have $h_j^\nu(y, \xi_j^\nu) = 0$ and $\nabla_\xi h_j^\nu(y, \xi) = \langle \varphi_{\xi\xi}''(y, \xi), \xi_j^\nu \rangle$. Therefore, we get

$$\nabla_\xi h_j^\nu(y, \xi_j^\nu) = 0$$

by Euler's formula. Writing the Taylor expansion of $h_j^\nu(y, \xi)$ with respect to ξ at ξ_j^ν , we obtain

$$|h_j^\nu(y, \xi)| \leq M \langle y \rangle |\text{vers}(\xi) - \xi_j^\nu|^2 \leq M 2^{-j},$$

as desired. \square

Proposition 3.5. $\|\mathcal{T}a_Q\|_{L^1(\mathcal{N}_Q)} \leq C$ with C independent of a_Q .

Proof. First we observe that operator $M_{\frac{n}{2}}\mathcal{T}(1 - \Delta)^{\frac{n-1}{4}}$ is a Fourier integral operator with the same phase and same properties of the amplitude as those of \mathcal{T} in view of the global calculus in [31]. Consequently, operator $M_{\frac{n}{2}}\mathcal{T}(1 - \Delta)^{\frac{n-1}{4}}$ is bounded on $L^2(\mathbb{R}^n)$ in view of the L^2 -boundedness theorems in [2] under assumption (2a) and in [28] under assumptions (2b) and (2c). Writing $p_n = \frac{2n}{2n-1}$ and recalling (3.10), we have

$$\begin{aligned} \|\mathcal{T}a_Q\|_{L^1(\mathcal{N}_Q)} &= \left\| M_{-\frac{n}{2}} \left[M_{\frac{n}{2}}\mathcal{T}(1 - \Delta)^{\frac{n-1}{4}} \right] (1 - \Delta)^{-\frac{n-1}{4}} a_Q \right\|_{L^1(\mathcal{N}_Q)} \\ &\leq \left(\int_{\substack{\mathcal{N}_Q \\ \langle x \rangle \sim \langle y \rangle}} \langle x \rangle^{-n} dx \right)^{\frac{1}{2}} \left\| \left[M_{\frac{n}{2}}\mathcal{T}(1 - \Delta)^{\frac{n-1}{4}} \right] \left[(1 - \Delta)^{-\frac{n-1}{4}} a_Q \right] \right\|_{L^2(\mathbb{R}^n)} \\ &\leq C_1 \left(\langle y_0 \rangle^{-n} |Q|^{\frac{1}{n}} \langle y_0 \rangle^{n-1} \right)^{\frac{1}{2}} \left\| (1 - \Delta)^{-\frac{n-1}{4}} a_Q \right\|_{L^2(\mathbb{R}^n)} \\ &\leq C_2 |Q|^{\frac{1}{2n}} \|a_Q\|_{L^{p_n}(\mathbb{R}^n)} \leq C |Q|^{\frac{1}{2n}} |Q|^{-\frac{1}{2n}} = C, \end{aligned}$$

with a constant C independent of a_Q , in view of the Hardy-Littlewood-Sobolev inequality

$$\left\| (1 - \Delta)^{-\frac{n-1}{4}} a_Q \right\|_{L^2(\mathbb{R}^n)} \leq \tilde{C} \|a_Q\|_{L^{p_n}(\mathbb{R}^n)},$$

and since, obviously,

$$\|a_Q\|_{L^{p_n}(\mathbb{R}^n)} \leq |Q|^{\frac{1}{p_n}-1} = |Q|^{-\frac{1}{2n}}.$$

\square

We will now prove the estimate

$$(3.11) \quad \|\mathcal{T}a_Q\|_{L^1(\mathbb{R}^n \setminus \mathcal{N}_Q)} \leq C$$

off the exceptional set. We first introduce a dyadic decomposition, choosing function $\theta \in \mathcal{C}^\infty(\mathbb{R})$ such that $\text{supp } \theta \subset \left(\frac{1}{4}, 4\right)$ and such that for all $s > 0$ we have $\sum_{k \in \mathbb{Z}} \theta(2^{-k}s) = 1$. We now set

$$(3.12) \quad F_k(x, y) = \int_{\mathbb{R}^n} e^{i[\langle x, \xi \rangle - \varphi(y, \xi)]} b(x, y, \xi) \theta_k(\xi) d\xi,$$

where $\theta_k(\xi) = \theta(2^{-k}|\xi|)$. We can assume without loss of generality that $b(x, y, \xi) = 0$ for $|\xi| < 8$. Defining $\theta_0 = 1 - \sum_{k > 0} \theta_k$, we have $F = \sum_{k \geq 1} F_k$. Estimate (3.11) is then a

consequence of the following proposition, where we recall that j was introduced in a way that 2^{-j} is a sidelength of Q .

Proposition 3.6. *For all $y, y' \in Q$, $j, k \in \mathbb{N}$, $j \geq j_0$, we have*

$$(3.13) \quad \int_{\mathbb{R}^n \setminus \mathcal{N}_Q} |F_k(x, y)| dx \prec 2^{j-k} \text{ if } k > j,$$

$$(3.14) \quad \int_{\mathbb{R}^n} |F_k(x, y) - F_k(x, y')| dx \prec 2^{k-j} \text{ if } k \leq j.$$

Proof. For each $k \in \mathbb{N}$, let $\{\chi_k^\nu\}$, $\nu = 1, \dots, N(y, k)$, be a homogeneous partition of unity associated with the covering of the unit sphere with the balls $B(\xi_k^\nu, c_0 2^{-\frac{k}{2}} \langle y \rangle^{-\frac{1}{2}})$, as introduced above. Explicitly, we choose \mathcal{C}^∞ functions $\chi_k^\nu = \chi_k^\nu(y, \xi)$, homogeneous in ξ of degree 0, such that, for all $y \in \mathbb{R}^n$, we have

- $\chi_k^\nu(y, \text{vers}(\xi)) \equiv 1$ for $\text{vers}(\xi)$ in a neighbourhood of ξ_k^ν in \mathbb{S}^{n-1} ;
- $\chi_k^\nu(y, \xi) = 0$ if $|\text{vers}(\xi) - \xi_k^\nu| \geq c_0 2^{-\frac{k}{2}} \langle y \rangle^{-\frac{1}{2}}$;
- $\sum_\nu \chi_k^\nu = 1$;
- $|\partial^\gamma \chi_k^\nu(y, \xi)| \prec |\xi|^{-|\gamma|} (2^k \langle y \rangle)^{\frac{|\gamma|}{2}}$ for all multi-indices $\gamma \in \mathbb{Z}_+^n$.

We now define

$$F_k^\nu(x, y) = \int_{\mathbb{R}^n} e^{i[\langle x, \xi \rangle - \varphi(y, \xi)]} b_k^\nu(x, y, \xi) d\xi,$$

where

$$b_k^\nu(x, y, \xi) = b(x, y, \xi) \theta_k(\xi) \chi_k^\nu(y, \xi).$$

Set also

$$r_k^\nu(y, \xi) = \varphi(y, \xi) - \langle \nabla_\xi \varphi(y, \xi_k^\nu), \xi \rangle$$

implying that

$$\nabla_\xi r_k^\nu(y, \xi) = \nabla_\xi \varphi(y, \xi) - \nabla_\xi \varphi(y, \xi_k^\nu),$$

and let $D_k^\nu = \langle \nabla_\xi, \xi_k^\nu \rangle$, $\nu = 1, \dots, N(k, y)$. Clearly, by definition of r_k^ν and homogeneity of φ , we have $r_k^\nu(y, \xi_k^\nu) = 0$ and $\nabla_\xi r_k^\nu(y, \xi_k^\nu) = 0$. Since, again by homogeneity,

$$\begin{aligned} (D_k^\nu r_k^\nu)(y, \xi) &= D_k^\nu \varphi(y, \xi) - \varphi(y, \xi_k^\nu) \implies (D_k^\nu r_k^\nu)(y, \xi_k^\nu) = 0, \\ (\nabla_\xi D_k^\nu r_k^\nu)(y, \xi) &= D_k^\nu \nabla_\xi \varphi(y, \xi) \implies (\nabla_\xi D_k^\nu r_k^\nu)(y, \xi_k^\nu) = 0, \end{aligned}$$

by induction we also see that, for all $N \in \mathbb{N}$, we have

$$(3.15) \quad [(D_k^\nu)^N r_k^\nu](y, \xi_k^\nu) = 0, \quad [\nabla_\xi (D_k^\nu)^N r_k^\nu](y, \xi_k^\nu) = 0.$$

Writing the Taylor expansion in ξ of r_k^ν centred in ξ_k^ν , (3.15) implies that, for all $N \in \mathbb{N}$, on $\text{supp}(b_k^\nu)$ we have

$$(3.16) \quad [(D_k^\nu)^N r_k^\nu](y, \xi) \prec |\xi|^{1-N} \langle y \rangle |\text{vers}(\xi) - \xi_k^\nu|^2 \prec 2^{k(1-N)} 2^{-k} = 2^{-kN}.$$

On the other hand, for the “transversal derivatives” with $|\gamma| \geq 1$ we have, on $\text{supp}(b_k^\nu)$,

$$(3.17) \quad D_\xi^\gamma r_k^\nu(y, \xi) \prec |\xi|^{1-|\gamma|} \langle y \rangle \prec 2^{-k(|\gamma|-1)} \langle y \rangle \prec 2^{-k \frac{|\gamma|}{2}} \langle y \rangle.$$

Indeed, first we recall that on $\text{supp}(b_k^\nu)$, $|\xi|$ is equivalent to 2^k . For $|\gamma| \geq 2$, we then have $|\xi|^{1-|\gamma|} \prec 2^{k(1-|\gamma|)} \leq 2^{-k \frac{|\gamma|}{2}}$ and hence also (3.17). For $|\gamma| = 1$, the first

derivatives are actually bounded by $2^{-\frac{k}{2}} \langle y \rangle^{\frac{1}{2}}$, since by $\nabla_{\xi} r_k^{\nu}(y, \xi_k^{\nu}) = 0$ and Taylor expansion we have

$$(\partial_{\xi_j} r_k^{\nu})(y, \xi) \prec \langle y \rangle |\xi - \xi_k^{\nu}| \prec 2^{-\frac{k}{2}} \langle y \rangle^{\frac{1}{2}}.$$

Consequently, one can readily check that on $\text{supp}(b_k^{\nu})$, we have estimate

$$(3.18) \quad D_{\xi}^{\gamma} e^{ir_k^{\nu}(y, \xi)} \prec 2^{-k \frac{|\gamma|}{2}} \langle y \rangle^{\frac{|\gamma|}{2}}.$$

Performing a rotation² $\xi = C\tilde{\xi}$, we can simplify notation and assume $\xi_k^{\nu} = (1, 0, \dots, 0)$, $\Pi_k^{\nu}(\xi) = (0, \xi')$. Rewriting $F_k^{\nu}(x, y)$ as

$$(3.19) \quad F_k^{\nu}(x, y) = \int_{\mathbb{R}^n} e^{i\langle x - \nabla_{\xi} \varphi(y, \xi), \xi_k^{\nu} \rangle} \tilde{b}_k^{\nu}(x, y, \xi) d\xi,$$

where $\tilde{b}_k^{\nu}(x, y, \xi) = e^{ir_k^{\nu}(y, \xi)} b_k^{\nu}(x, y, \xi)$, we observe that the derivatives in the ξ_1 (“radial”) direction of χ_k^{ν} vanish identically, so that, defining the selfadjoint operator L_k^{ν} as

$$L_k^{\nu} = \left(I - 2^{2k} \frac{\partial^2}{\partial \xi_1^2} \right) (I - 2^k \langle y \rangle^{-1} \langle \nabla_{\xi'}, \nabla_{\xi'} \rangle),$$

(3.16), (3.18), the properties of χ_k^{ν} , the definition of θ_k and the hypotheses on φ and b imply, for all $N \in \mathbb{N}$, that we have

$$(3.20) \quad [(L_k^{\nu})^N \tilde{b}_k^{\nu}](x, y, \xi) \prec 2^{-k \frac{n-1}{2}} \langle y \rangle^{-\frac{n}{2}}.$$

Repeated integrations by parts allow to write

$$F_k^{\nu}(x, y) = H_{k,N}^{\nu}(x, y) \int_{\mathbb{R}^n} e^{i\langle x - \nabla_{\xi} \varphi(y, \xi_k^{\nu}), \xi \rangle} [(L_k^{\nu})^N \tilde{b}_k^{\nu}](x, y, \xi) d\xi,$$

with

$$H_{k,N}^{\nu}(x, y) = (1 + |2^k(x - \nabla_{\xi} \varphi(y, \xi_k^{\nu}))_1|^2)^{-N} \left(1 + |2^{\frac{k}{2}} \langle y \rangle^{-\frac{1}{2}} (x - \nabla_{\xi} \varphi(y, \xi_k^{\nu}))'|^2 \right)^{-N}.$$

Since

$$\text{vol}_{\xi}(\text{supp}(\tilde{b}_k^{\nu})) \prec 2^k (2^k \cdot 2^{-\frac{k}{2}} \langle y \rangle^{-\frac{1}{2}})^{n-1} = 2^{k \frac{n+1}{2}} \langle y \rangle^{-\frac{n-1}{2}},$$

by (3.20) it follows that

$$(3.21) \quad |F_k^{\nu}(x, y)| \prec H_{k,N}^{\nu}(x, y) 2^k \langle y \rangle^{-n+\frac{1}{2}}.$$

In $\mathbb{R}^n \setminus \mathcal{N}_Q$, we must have either $|2^k(x - \nabla_{\xi} \varphi(y, \xi_k^{\nu}))_1| \succ 2^{k-j}$ or $|2^{\frac{k}{2}} \langle y \rangle^{-\frac{1}{2}} (x - \nabla_{\xi} \varphi(y, \xi_k^{\nu}))'| \succ 2^{\frac{k-j}{2}}$. Since, obviously, $H_{k,N}^{\nu} = H_{k,N-N'}^{\nu} \cdot H_{k,N'}^{\nu}$ for any $N, N' \in \mathbb{N}$ such that $N > N'$, then, for any $k > j$, we can estimate

$$(3.22) \quad \int_{\mathbb{R}^n} H_{k,N}^{\nu}(x, y) dx \leq C_{N-N'} 2^{-k} 2^{-k \frac{n-1}{2}} \langle y \rangle^{\frac{n-1}{2}} 2^{-N'(k-j)},$$

which implies, together with (3.21), that

$$(3.23) \quad \int_{\mathbb{R}^n} |F_k^{\nu}(x, y)| dx \prec 2^{j-k} 2^{-k \frac{n-1}{2}} \langle y \rangle^{-\frac{n}{2}}.$$

²Note that all the symbol estimates for θ_k , χ_k^{ν} , r_k^{ν} , φ , and b hold unchanged for fixed y , since all the entries of C are bounded, in view of $A \in O(n)$.

Now (3.13) follows from (3.23), by summing over $\nu = 1, \dots, N(y, k)$. Owing to

$$\begin{aligned} \int_{\mathbb{R}^n} |F_k(x, y) - F_k(x, y')| dx &\leq \sum_{\nu} \int_{\mathbb{R}^n} |F_k^{\nu}(x, y) - F_k^{\nu}(x, y')| dx \\ &\leq |y - y'| \sum_{\nu} \int_{\mathbb{R}^n} \sup_{y \in Q} |\nabla_y F_k^{\nu}(x, y)| dx \prec 2^{-j} \sum_{\nu} \int_{\mathbb{R}^n} \sup_{y \in Q} |\nabla_y F_k^{\nu}(x, y)| dx, \end{aligned}$$

estimate (3.14) would follow from

$$(3.24) \quad \int_{\mathbb{R}^n} \sup_{y \in Q} |\nabla_y F_k^{\nu}(x, y)| dx \prec 2^k \cdot 2^{-k \frac{n-1}{2}} \langle y_0 \rangle^{-\frac{n}{2}}.$$

Now, (3.24) indeed holds true, since $\nabla_y F_k^{\nu}(x, y)$ can be written in the form (3.19) with

$$\tilde{a}_k^{\nu}(x, y, \xi) = \nabla_y \tilde{b}_k^{\nu}(x, y, \xi) - i \tilde{b}_k^{\nu}(x, y, \xi) \cdot \nabla_y \varphi(y, \xi)$$

in place of $\tilde{b}_k^{\nu}(x, y, \xi)$, and $\tilde{a}_k^{\nu}(x, y, \xi)$ has the same properties of $\tilde{b}_k^{\nu}(x, y, \xi)$ with order in ξ increased by one unit. It is then possible to repeat the same argument used in the proof of (3.23), and to sum over $\nu = 1, \dots, N(y, k)$, recalling that $\langle y \rangle \sim \langle y_0 \rangle$ for $y \in Q$. \square

Conclusion of the proof of (3.11): by properties (1), (2) and (3) of a_Q and Proposition 3.6, denoting by \mathcal{T}_k the operator with kernel F_k defined in (3.12), we have

$$\begin{aligned} \|\mathcal{T}a_Q\|_{L^1(\mathbb{R}^n \setminus \mathcal{N}_Q)} &\leq \sum_{k \geq 0} \|\mathcal{T}_k a_Q\|_{L^1(\mathbb{R}^n \setminus \mathcal{N}_Q)} \\ &\leq \sum_{0 \leq k \leq j} \int_{\mathbb{R}^n} \left| \int_Q [F_k(x, y) - F_k(x, y')] a_Q(y) dy \right| dx \\ &\quad + \sum_{k > j} \int_{\mathbb{R}^n \setminus \mathcal{N}_Q} \left| \int_Q F_k(x, y) a_Q(y) dy \right| dx \\ &\leq \sum_{0 \leq k \leq j} \int_Q \left[\int_{\mathbb{R}^n} |F_k(x, y) - F_k(x, y')| dx \right] |a_Q(y)| dy \\ &\quad + \sum_{k > j} \int_Q \left[\int_{\mathbb{R}^n \setminus \mathcal{N}_Q} |F_k(x, y)| dx \right] |a_Q(y)| dy \\ &\leq \frac{C}{3} \left(\sum_{0 \leq k \leq j} 2^{k-j} + \sum_{k > j} 2^{j-k} \right) \leq C, \end{aligned}$$

with C independent of a_Q , as claimed.

Remark 3.7. We note that statements of Propositions 3.5 and 3.6 remain true if operator \mathcal{T} satisfies assumptions of Theorem 2.3 only with $m \leq -(n-1)/2$.

4. PROOF OF THEOREM 2.6

A preliminary result to be proven is the following

Proposition 4.1 ($L^p(\mathbb{R}^n)$ -boundedness of localised Fourier integral operators). *Assume the hypotheses in Theorem 2.6 and let $\tilde{\psi} \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be supported in the shell $2^{-2} \leq |x| \leq 2^2$. Then we have, for $k \geq 1$,*

$$\|\tilde{\psi}(2^{-k}x)Af\|_{L^p} \leq C\|f\|_{L^p},$$

where the constant C depends only on $\tilde{\psi}$, on upper bounds for a finite number of the constants in the estimates satisfied by a and φ , and on the lower bound δ for the determinant of the mixed Hessian of φ .

Proof. We can write

$$\tilde{\psi}(2^{-k}x)A = U_{2^{-k}}A'_kU_{2^k},$$

where $U_\lambda f(x) = f(\lambda x)$, $\lambda \neq 0$, is the dilation operator and

$$A'_k f(x) = \int_{\mathbb{R}^n} e^{i\varphi(2^k x, 2^{-k}\xi)} \tilde{\psi}(x) a(2^k x, 2^{-k}\xi) \widehat{f}(\xi) d\xi.$$

Hence it suffices to prove the desired conclusion with A'_k in place of $\tilde{\psi}(2^{-k}x)A$. It follows from the estimates satisfied by φ and the fact that $|x| \sim 1$ on the support of $\tilde{\psi}$ that, there,

$$|\partial_x^\alpha \partial_\xi^\beta (\varphi(2^k x, 2^{-k}\xi))| \leq M_{\alpha,\beta} |\xi|^{1-|\beta|},$$

(in fact, $\langle 2^k x \rangle \sim 2^k$ on the support of $\tilde{\psi}$). Moreover, we immediately have

$$(4.1) \quad \left| \det \left(\frac{\partial^2 (\varphi(2^k x, 2^{-k}\xi))}{\partial \xi_j \partial x_l} \right) \right| > \delta > 0.$$

Similarly, one sees that³, on the support of $\tilde{\psi}$, we have

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta (a(2^k x, 2^{-k}\xi))| &= 2^{k(|\alpha|-|\beta|)} |(\partial_x^\alpha \partial_\xi^\beta a)(2^k x, 2^{-k}\xi)| \\ &\leq C_{\alpha,\beta} 2^{k(|\alpha|-|\beta|)} \langle 2^k x \rangle^{m-|\alpha|} \langle 2^{-k}\xi \rangle^{\mu-|\beta|} \\ &\leq C_{\alpha,\beta} 2^{k(|\alpha|-|\beta|)} \langle 2^k x \rangle^{m_p-|\alpha|} \langle 2^{-k}\xi \rangle^{m_p-|\beta|} \\ &\leq C_{\alpha,\beta} 2^{k(|\alpha|-|\beta|+m_p-|\alpha|-m_p+|\beta|)} \langle \xi \rangle^{m_p-|\beta|} \\ &= C_{\alpha,\beta} \langle \xi \rangle^{m_p-|\beta|}, \end{aligned}$$

where we have set $m_p = -(n-1) \left| \frac{1}{p} - \frac{1}{2} \right| \geq m, \mu$.

We have then showed that the operators A'_k satisfy the assumptions of Seeger-Sogge-Stein's theorem, uniformly with respect to $k \in \mathbb{N}$: an application of that theorem concludes the proof⁴. \square

³Precisely, to verify this last estimate, distinguish the case $|\xi| \leq 2^k$ (which implies $\langle 2^{-k}\xi \rangle \sim 1$, $\langle \xi \rangle \prec 2^k \Rightarrow 1 \prec 2^{k(-m_p+|\beta|)} \langle \xi \rangle^{m_p-|\beta|}$) and the case $|\xi| \geq 2^k$ (which implies $\langle 2^{-k}\xi \rangle \sim |2^{-k}\xi|$, $\langle \xi \rangle \sim |\xi|$).

⁴Indeed, it suffices to observe that the amplitudes of the A'_k , $k \in \mathbb{N}$, are compactly supported and all the other requirements of the Seeger-Sogge-Stein's theorem are fulfilled; moreover, the constant in the boundedness estimate of the aforementioned Theorem depends only on upper bounds for a finite number of the constants in the estimates satisfied by the phase and amplitude functions, and a lower bound for the mixed Hessian of the phase.

We then make use of a Littlewood–Paley partition of unity $\{\psi_k\}$, $k \in \mathbb{Z}_+$, such that $\psi_0 \in C_0^\infty(\mathbb{R}^n)$, $\psi_k(x) = \psi(2^{-k}x)$, $k \geq 1$, $\text{supp } \psi \subset \{x \in \mathbb{R}^n : 2^{-1} \leq |x| \leq 2\}$, and write the operator A of (2.12) as

$$(4.2) \quad A = \psi_0 A + \sum_{k=1}^{\infty} \psi_k A.$$

The operator $\psi_0 A$ is L^p -bounded by the Seeger-Sogge-Stein's theorem [32], so we only treat the second term in (4.2), namely, the sum over $k \geq 1$, writing

$$\sum_{k=1}^{\infty} \psi_k A = \sum_{k=1}^{\infty} \sum_{k'=0}^{\infty} \psi_k A \psi_{k'}.$$

The functions ψ_k , $k \geq 1$, can be interpreted as SG pseudo-differential operators, so that it is possible to use the composition formulae of a SG Fourier integral operator with a SG pseudo-differential operator, see [12] or [30, 31]. Splitting the asymptotic expansion of the amplitude of the composed operator into the sum of the terms from order (m, μ) to order $(m-3, \mu-3)$ and of the corresponding remainder, we write

$$(4.3) \quad \psi_k A \psi_{k'} = A_{k,k'} + 2^{-k-k'} R_{k,k'}.$$

Actually, we can compose the operators in (4.3) on the left with the multiplication by $\tilde{\psi}_k(x) := \tilde{\psi}(2^{-k}x)$, and on the right with the multiplication by $\tilde{\psi}_{k'}(x)$, for a suitable cut off $\tilde{\psi}$, so that $\tilde{\psi}_k \psi_k = \psi_k$. This does not affect the left-hand side and we find

$$\psi_k A \psi_{k'} = \tilde{\psi}_k A_{k,k'} \tilde{\psi}_{k'} + 2^{-k-k'} \tilde{\psi}_k R_{k,k'} \tilde{\psi}_{k'},$$

with Fourier integral operators $A_{k,k'}$ and $R_{k,k'}$, with amplitudes in $S^{m,\mu}$ and in $S^{m,\mu-2}$, respectively (*uniformly with respect to k, k'*). Note also that, in view of the properties of the Littlewood–Paley partition of unity and the formula for the asymptotic expansion of the amplitude of the composition of a pseudo-differential operator and a Fourier integral operator, $|k - k'| > N$ implies $A_{k,k'} \equiv 0$, for some fixed $N > 0$. Proposition 4.1 applied with $A_{k,k'}$ in place of A and $\tilde{\psi}_{k'} f$ in place of f , together with the properties of the dyadic decomposition $\{\psi_k\}$, $k \in \mathbb{Z}_+$, gives the desired estimate

for the operator $\sum_{k=1}^{\infty} \sum_{k'=0}^{\infty} \tilde{\psi}_k A_{k,k'} \tilde{\psi}_{k'}$:

$$\begin{aligned} & \left\| \sum_{k=1}^{\infty} \sum_{k' \geq 0, |k'-k| \leq N} \tilde{\psi}_k A_{k,k'} \tilde{\psi}_{k'} f \right\|_{L^p}^p \prec \sum_{k=1}^{\infty} \left\| \sum_{k' \geq 0, |k'-k| \leq N} \tilde{\psi}_k A_{k,k'} \tilde{\psi}_{k'} f \right\|_{L^p}^p \\ & \prec \sum_{k=1}^{\infty} \sum_{k' \geq 0, |k'-k| \leq N} \|\tilde{\psi}_k A_{k,k'} \tilde{\psi}_{k'} f\|_{L^p}^p \\ & \prec \sum_{k=1}^{\infty} \sum_{k' \geq 0, |k'-k| \leq N} \|\tilde{\psi}_{k'} f\|_{L^p}^p \leq (2N+1) \sum_{k'=0}^{\infty} \|\tilde{\psi}_{k'} f\|_{L^p}^p \prec \|f\|_{L^p}^p, \end{aligned}$$

where we used $\sum_{k'=0}^{\infty} \|\tilde{\psi}_{k'} f\|_{L^p}^p \prec \|f\|_{L^p}^p$, $\left\| \sum_{k=1}^{\infty} \tilde{\psi}_k u_k \right\|_{L^p}^p \prec \sum_{k=1}^{\infty} \|\tilde{\psi}_k u_k\|_{L^p}^p$, which hold for arbitrary $f, u_k \in L^p(\mathbb{R}^n)$, $k \geq 1$. A similar argument allows to estimate

$$\left\| \sum_{k=1}^{\infty} \sum_{k'=0}^{\infty} 2^{-k-k'} \tilde{\psi}_k R_{k,k'} \tilde{\psi}_{k'} f \right\|_{L^p} \leq \sum_{k=1}^{\infty} \sum_{k'=0}^{\infty} 2^{-k-k'} \|\tilde{\psi}_k R_{k,k'} \tilde{\psi}_{k'} f\|_{L^p}.$$

Indeed, again by Proposition 4.1 applied with $R_{k,k'}$ in place of A , and $\tilde{\psi}_{k'} f$ in place of f , we see that the right hand side is

$$\prec \sum_{k=1}^{\infty} \sum_{k'=0}^{\infty} 2^{-k-k'} \|\tilde{\psi}_{k'} f\|_{L^p} = \sum_{k'=0}^{\infty} 2^{-k'} \|\tilde{\psi}_{k'} f\|_{L^p},$$

and, by an application of Hölder's inequality, the last expression is dominated by

$$\prec \left(\sum_{k'=0}^{\infty} \|\tilde{\psi}_{k'} f\|_{L^p}^p \right)^{1/p} \prec \|f\|_{L^p}.$$

5. APPLICATIONS TO HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

In this section we briefly give applications of the obtained results to the solutions of the Cauchy problem for strictly hyperbolic partial differential equations. Theorems 5.1 and 5.2 describe the loss of regularity and weight for solutions. We restrict the statements to large frequencies, also because it is known that different phenomena may occur for small frequencies.

Let us first look at the equation of the first order

$$(5.1) \quad \begin{cases} (D_t + a(t, x, D_x)u(t, x) = 0, & t \neq 0, \\ u|_{t=0} = f(x). \end{cases}$$

As usual, $D_t = -i\partial_t$ and $D_{x_j} = -i\partial_{x_j}$. We assume that the symbol $a(t, x, \xi)$ is a classical symbol of order one depending smoothly on t, x and ξ . The strict hyperbolicity means that the principal symbol $a_1(t, x, \xi)$ is real-valued.

The result of [32] states that if $f \in W^{p, \alpha + (n-1)|1/p-1/2|}(\mathbb{R}^n)$, for some $\alpha \in \mathbb{R}$, it follows that the solution satisfies $u(t, \cdot) \in W^{p, \alpha}(\mathbb{R}^n)$ locally, $1 < p < \infty$. Moreover, this order is sharp when for every t in the complement of a discrete set in \mathbb{R} , a is elliptic in ξ .

We now give a global version of this result. It follows from [20, Section 4, Ch. 10] that modulo a smooth bounded function, for sufficiently small times, the solution $u(t, x)$ to (5.1) can be constructed as a Fourier integral operator in the form (2.1). If we assume that a is a classical symbol with real-valued principal part such that

$$(5.2) \quad |\partial_t^k \partial_x^\beta \partial_\xi^\alpha a(t, x, \xi)| \leq C_{k\alpha\beta} \langle \xi \rangle^{1-|\alpha|}$$

holds for all $x, \xi \in \mathbb{R}^n$, all $t \in [0, T]$ for some $T > 0$, and all k, α, β , with constants $C_{k\alpha\beta}$ independent of t, x, ξ , then the phase and the amplitude of the propagator satisfy assumption (2a) of Theorem 2.2. Thus, cutting off low frequencies, we obtain

Theorem 5.1. *Let the symbol $a(t, x, \xi)$ satisfy conditions (5.2). Let $1 < p < \infty$, and let $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 1$ for $|\xi| \leq \varepsilon$, for some sufficiently large $\varepsilon > 0$. If f is such that $\langle x \rangle^{n|1/p-1/2|} f(x) \in W^{p, (n-1)|1/p-1/2|}(\mathbb{R}^n)$, then for each $t \in [0, T]$,*

the solution $u(t, x)$ of the Cauchy problem (5.1) satisfies $(1 - \chi(D))u(t, \cdot) \in L^p(\mathbb{R}^n)$. Moreover, for every $\alpha \in \mathbb{R}$ and $m \in \mathbb{R}$, there is $C_T > 0$ such that we have the estimate

$$\|\langle x \rangle^m (1 - \chi(D))u(t, \cdot)\|_{W^{p, \alpha}(\mathbb{R}^n)} \leq C_T \|\langle x \rangle^{m+n|1/p-1/2|} f\|_{W^{p, \alpha+(n-1)|1/p-1/2|}(\mathbb{R}^n)},$$

for all $t \in [0, T]$ and all f such that the right hand side norm is finite.

In the case when the symbol $a(t, x, \xi)$ has an SG-behavior, we have an improvement. Assume that a is a classical symbol with the real-valued principal part such that

$$(5.3) \quad |\partial_t^k \partial_x^\beta \partial_\xi^\alpha a(t, x, \xi)| \leq C_{k\alpha\beta} \langle x \rangle^{-|\beta|} \langle \xi \rangle^{1-|\alpha|}$$

holds for all $x, \xi \in \mathbb{R}^n$, all $t \in [0, T]$ for some $T > 0$, and all k, α, β , with constants $C_{k\alpha\beta}$ independent of t, x, ξ . Then it was shown in [13] that the solution to the Cauchy problem (5.1) can be written in the form (2.1), with phase and amplitude satisfying conditions of Theorem 2.6. Thus, we obtain

Theorem 5.2. *Let the symbol $a(t, x, \xi)$ satisfy conditions (5.3). Let $1 < p < \infty$, and let $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi(\xi) = 1$ for $|\xi| \leq \varepsilon$, for some sufficiently large $\varepsilon > 0$. If f is such that $\langle x \rangle^{(n-1)|1/p-1/2|} f(x) \in W^{p, (n-1)|1/p-1/2|}(\mathbb{R}^n)$, then for each $t \in [0, T]$, the solution $u(t, x)$ of the Cauchy problem (5.1) satisfies $(1 - \chi(D))u(t, \cdot) \in L^p(\mathbb{R}^n)$. Moreover, for every $\alpha \in \mathbb{R}$ and $m \in \mathbb{R}$, there is $C_T > 0$ such that we have the estimate*

$$\|\langle x \rangle^m (1 - \chi(D))u(t, \cdot)\|_{W^{p, \alpha}(\mathbb{R}^n)} \leq C_T \|\langle x \rangle^{m+(n-1)|1/p-1/2|} f\|_{W^{p, \alpha+(n-1)|1/p-1/2|}(\mathbb{R}^n)},$$

for all $t \in [0, T]$ and all f such that the right hand side norm is finite.

We note that following [20] and [13], similar conclusions hold for higher order equations under appropriate conditions on lower order terms.

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